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Finite Bases for Flat Graph Algebras[1]

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In this article the classification of finite flat graph algebras which have finite equational bases is given in terms of omitted induced subgraphs. The result is related to an earlier result obtained for finite graph algebras by Baker, McNulty, and Werner.

Key Words: graph algebras; flat algebras; equational theory; finitely based algebras.

1. INTRODUCTION

One of the questions arising naturally in the study of familiar algebraic structures, such as groups, rings, and lattices, is, "Can the set of all identities valid in the class of algebras under consideration be derived from a particular finite set of familiar identities; i.e, can the class in question be axiomatized in a finite way?" For a significant number of such classes this question yields an affirmative answer. For example, the class of all groups has a familiar axiomatization using the three identities

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad x \cdot 1 = x, \qquad x \cdot x^{-1} = 1,$$

in the standard language $\{\cdot, {}^{-1}, 1\}$. Another question that merits investigation in its own right is whether a class consisting of a single structure can be finitely axiomatized or whether the structure in question is *finitely based*. We will give a more precise formulation of these notions shortly.

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Finite basis questions in universal algebra have a long history and have fostered much interaction between algebra and mathematical logic. Originating in the work of Garrett Birkhoff and Roger Lyndon in the 1950s, through deep results of Oates and Powell concerning equational axiomatizability of finite groups, interest in the questions of finite axiomatizability of algebraic systems was recently invigorated by a series of results by Ralph McKenzie, who answered several long-standing conjectures in universal algebra, the most famous of which is undoubtedly his negative answer to Tarski's finite basis conjecture.

Virtually all finite algebras tracing their heritage back to the nineteenth century, such as groups, rings, and Boolean algebras, have finite equational bases, even though the proofs of these facts are far from being obvious.

One of the principal goals of equational logic, as a branch of universal algebra, was to provide an understanding of more general classes of algebras that have finitely axiomatizable equational theories. This is even more important in the light of McKenzie's negative answer to Tarski's finite basis problem, proving that there is no algorithmic characterization of *all* finitely axiomatizable finite algebras. Therefore, it seems reasonable to take another route to seek wide domains where such characterization is still possible.

The investigation carried out in this article was motivated by recent results of Willard [12] on congruence meet–semidistributive varieties of algebras, where it is shown that the question of finite equational bases for a finite algebra is, in fact, a question about its residual character.

Whereas earlier proofs relied heavily on the syntactic analysis of equational derivability in the class of algebras under investigation, our methods are of a rather different nature: they involve definability of certain compatible equivalence relations on the algebra as well as finite axiomatizability of a significantly smaller subclass.

We start our exposition by reviewing several basic concepts of general algebra which will play a fundamental role in what follows. We presume some basic familiarity with elementary model theory for first-order logic, for which the reader is referred to [3].

Given a first-order language L containing only function symbols, an *algebra* in L (or an L *algebra*) is a first-order structure $\mathbf{A} = \langle A, F \rangle$ consisting of a nonempty set A and a family F of operations on A indexed by the symbols from L in the usual way.

By an *equation* in L we understand the equality of two terms $s = t$ from L.

Given a class of L algebras $\mathcal{K}$, the *equational theory* of $\mathcal{K}$ is the set of all L equations true in every algebra from $\mathcal{K}$. Via an obvious identification, the equational theory of $\mathcal{K}$ can be identified with a fragment of the theory of all universal first-order sentences true in $\mathcal{K}$.

For a class of algebras $\mathcal{K}$ and Σ_0, a set of equations true in every algebra of $\mathcal{K}$, we say that Σ_0 is an *equational basis* for $\mathcal{K}$ (or that Σ_0 *axiomatizes* $\mathcal{K}$) if

$$\mathbf{A} \in \mathcal{K} \quad \text{if and only if} \quad \mathbf{A} \models \Sigma_0.$$

If Σ_0 is finite, the class $\mathcal{K}$ (and its equational theory) is said to be *finitely based*.

One of the fundamental questions in universal algebra is to determine for what finite algebras $\mathbf{A}$ the class $\mathcal{K} = \{\mathbf{A}\}$ is finitely based. Such algebras are said to be *finitely based*, and the examples of such algebras of more classical provenance are abundant in number; among others, they include all finite groups (Oates-Williams and Powell), finite associative rings (Kruse and Lvov), lattices (McKenzie), Boolean algebras, and commutative semigroups. For more information on the finite equational basis problem for algebras, see [8].

A *variety* of algebras, in a language L, is any class of L algebras axiomatized by some set of L equations. It turns out that varieties are precisely those classes of algebras that are closed under the formation of homomorphic images, direct (Cartesian) products, and substructures. This can be expressed in the following way: given a class $\mathcal{K}$ of L algebras, the smallest variety containing $\mathcal{K}$ (denoted by $\mathbf{V}(\mathcal{K})$) is the class

$$\mathbf{V}(\mathcal{K}) = \mathbf{HSP}(\mathcal{K}),$$

where **H, S,** and **P** are closure operators of taking homomorphic images, subalgebras, and Cartesian products, respectively. An algebra is said to be *locally finite*, if every finitely generated subalgebra is also finite; a variety is said to be locally finite if every one of its members is a locally finite algebra.

DEFINITION 1. A finite algebra $\mathbf{A}$ with finitely many fundamental operations is said to be *inherently nonfinitely based* if $\mathbf{A}$ does not belong to any locally finite finitely based variety.

Clearly, any finite algebra which is inherently nonfinitely based is nonfinitely based.

One of the fundamental notions studied in the context of universal algebra is the one of a *congruence*. Given an algebra $\mathbf{A} = \langle A, F\rangle$, a congruence ϕ on $\mathbf{A}$ is any equivalence relation on the universe A, which is compatible with all fundamental operations from F; i.e., if $f \in F$ is an n-ary operation of $\mathbf{A}$ and $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A$ such that

$$\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \dots, \langle a_n, b_n\rangle \in \phi,$$

then

$$\langle f(a_1, a_2, \dots, a_n), f(b_1, b_2, \dots, b_n)\rangle \in \phi.$$

It is easily seen that the collection of all congruences of an algebra **A**, Con(**A**), is a lattice under inclusion.

If $S \subseteq A \times A$, we denote the smallest congruence of **A** by $\mathrm{Cg}^{\mathbf{A}}(S)$; if $S = \{\langle a, b\rangle\}$, we simply write $\mathrm{Cg}^{\mathbf{A}}(a, b)$ for $\mathrm{Cg}^{\mathbf{A}}(S)$.

DEFINITION 2. Let **A** be a nontrivial algebra. **A** is said to be *subdirectly irreducible* if there exist elements $a, b \in A$, $a \neq b$, such that, for any pair of distinct elements $c, d \in A$,

$$\mathrm{Cg}^{\mathbf{A}}(a, b) \subseteq \mathrm{Cg}^{\mathbf{A}}(c, d).$$

The standard terminology for this pair $\langle a, b\rangle$ is a *critical pair*. It is easily seen that the congruence generated by a critical pair is the minimal nontrivial congruence of **A**.

If V is a variety of algebras, then V_{SI} will denote its class of subdirectly irreducible members. A fundamental result of universal algebra, owing to G. Birkhoff, states that two varieties are the same if they have the same subdirectly irreducible members.

DEFINITION 3. Given a nonempty set A, we say that a binary operation $\wedge$ is a *meet–semilattice operation* on A if it is associative, idempotent, and commutative, i.e., the structure $\langle A, \wedge\rangle$ satisfies the following equations:

1. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
2. $x \wedge x = x$.
3. $x \wedge y = y \wedge x$.

The notion of a meet–semilattice operation on A induces a partial ordering on the set, defined in the manner

$$a \leq b \quad \text{if and only if} \quad a \wedge b = a,$$

for all $a, b \in A$. If the induced partial ordering has the least element 0, then $\langle A, \wedge\rangle$ is said to be a *height*-1 meet–semilattice if, for every $a > 0$ in A, there is no element $b \in A$, such that $0 < b < a$.

DEFINITION 4. A *flat algebra* is an algebra **A** whose type includes a binary meet–semilattice operation $\wedge$ and a constant 0, such that

1. $\langle A, \wedge\rangle$ is a height-1 meet–semilattice with least element 0.
2. 0 is an absorbing element; that is, if f is an n-ary fundamental operation of **A** and $0 \in \{a_1, \ldots, a_n\}$, then $f(a_1, \ldots, a_n) = 0$.

In recent years flat algebras have drawn considerable interest on the part of general algebraists. The reason for this lies in the fact that expansions of these algebras were cleverly used by McKenzie [6, 7] to refute some of long-standing conjectures in universal algebra, such as the Quackenbush conjecture as well as Tarski's finite basis problem. The varieties generated by a finite flat algebra display another quite agreeable feature: there is a uniform way to describe subdirectly irreducible members of such a variety.

The issue of finite basis for flat algebras was addressed in [12], where it was proved, for instance, that every flat algebra which is of a finite residual character is finitely based. For a brief introduction to flat algebras, the reader is referred to [11] (Willard uses the terminology "M-algebra" instead).

Another essential ingredient in our exposition will be a flat graph algebra. A *graph*, for the purposes of this paper, is a relational structure $\mathbf{G} = \langle V, E \rangle$, consisting of a nonempty set V of *vertices* and a symmetric binary relation $E \subseteq V \times V$, called the set of *edges* of $\mathbf{G}$.

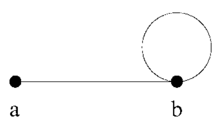
DEFINITION 5. Let $\mathbf{G} = \langle V, E \rangle$ be a graph, where V is the set of vertices of $\mathbf{G}$ and E is the set of undirected edges of $\mathbf{G}$. We define the *flat graph algebra* $\mathbf{G}^\wedge$, corresponding to $\mathbf{G}$, to be the flat algebra in the language $\{\circ, \wedge, 0\}$, whose universe is $A = V \cup \{0\}$, $0 \notin V$, whose binary operation $\circ$ on A is defined by

$$a \circ b = \begin{cases} a, & \text{if } \langle a, b \rangle \in E, \\ 0, & \text{otherwise,} \end{cases}$$

and whose smallest element under the meet–semilattice ordering, induced by $\wedge$, is 0. 0 is called the *absorbing element* for $\mathbf{G}^\wedge$.

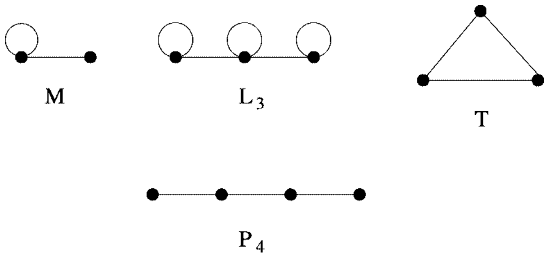
The definition of a flat graph algebra presents a variation of the notion of a graph algebra, first introduced and studied by Shallon in [10]. The relationship between the two notions is rather simple: a graph algebra is just a reduct to the language of $\{\circ, 0\}$ of a flat graph algebra.

In their paper [9], McNulty and Shallon give several examples of inherently nonfinitely based graph algebras, one of which is the so-called *Murskii groupoid*, the first known example of an inherently nonfinitely based algebra. This groupoid is a graph algebra based on the graph



The main result of our paper lies in the classification, in terms of omitted induced subgraphs, of those finite flat graph algebras which generate finitely based varieties, and is contained in the following theorem.

THEOREM 6. *If* $\mathbf{A}$ *is a finite flat graph algebra, which omits* $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, *and* $\mathbf{P}_4$, *then* $\mathbf{A}$ *has a finitely axiomatizable equational theory.*



The statement of this theorem, but not the proof, corresponds exactly to the theorem for graph algebras owing to Baker, McNulty, and Werner in [1].

In concluding this section we refer the reader to [3] for more information on the tools from universal algebra that will be used throughout the paper.

2. DEFINABLE ORDERED PRINCIPAL CONGRUENCES AND FINITE BASIS

In this section we give a general method for proving that a variety generated by flat algebras is finitely based. This method involves establishing two things: that the variety in question has the property that its class of subdirectly irreducible members is finitely axiomatizable and that the variety satisfies the definable ordered congruence property (a new property introduced in this paper; see Definition 7).

The main result of this section (Theorem 10) is closely related to a result of McKenzie [5], whereas our proof is a variation of the proof of McKenzie's theorem found in [3], [Theorem V.4.3].

DEFINITION 7. A *principal congruence formula* in the language L is a formula $\psi(x, y, u, v)$ of the form

$$\exists \bar{w}\left[x = t_1(z_1, \bar{w}) \ \& \ \left(\bigwedge_{1\le i<n} t_i(z_i', \bar{w}) = t_{i+1}(z_{i+1}, \bar{w})\right) \ \& \ t_n(z_n', \bar{w}) = y\right],$$

where $\bar{w}$ is an m-tuple of variables, $t_1, \dots, t_n$ are some $(m+1)$-ary L terms, and

$$\{z_i, z_i'\} = \{u, v\}$$

for all $1 \le i < n$.

Clearly, if $\mathbf{A}$ is an L algebra and $a, b, c, d \in A$, then

$$\langle a, b\rangle \in \mathrm{Cg}^{\mathbf{A}}(c, d) \quad \text{if and only if} \quad \mathbf{A} \models \psi(a, b, c, d)$$

for some principal congruence formula ψ.

DEFINITION 8. A variety V is said to have *definable principal congruences* if there is a finite collection Ψ of principal congruence formulas in the language of V such that for any $\mathbf{A} \in V$ and $a, b, c, d \in A$,

$$\langle a, b \rangle \in \mathrm{Cg}^{\mathbf{A}}(c, d) \quad \text{if and only if} \quad \mathbf{A} \models \psi(a, b, c, d)$$

for some $\psi \in \Psi$.

Since we are interested primarily in those varieties whose type contains a meet–semilattice operation, it is rather natural to weaken the property defined above so that it applies to a seemingly weaker class of congruences, namely, those generated by a pair of comparable elements. Thus, we arrive at the following definition.

DEFINITION 9. Let V be a variety such that there is a binary term operation in the language of V which induces a meet–semilattice operation in every algebra of V. We say that V has *definable ordered principal congruences* if there are finitely many principal congruence formulas which define the congruences of the form $\mathrm{Cg}^{\mathbf{A}}(a, b)$, where $b \leq a$, in every $\mathbf{A} \in V$.

Clearly, if a variety has definable principal congruences, it will also have definable principal ordered congruences. That the converse also holds was pointed out to us by Kearnes [4]. In the context of this article, however, the latter property is easier to establish and more convenient to work with, and we shall make extensive use of it.

Finally, let us point out before stating the main result of this section that among algebras with meet–semilattice operations the subdirectly irreducible algebras are the ones with an ordered critical pair.

THEOREM 10. *Let V be a variety with a binary term operation $\wedge$ which is a semilattice operation in every member of V. If V_{SI} is first-order definable and V has definable ordered principal congruences, then V is finitely based.*

Proof. First, note that if W is any variety in the language of V, in which $\wedge$ is a meet–semilattice operation, and $\mathbf{A} \in W$, any nontrivial principal congruence of $\mathbf{A}$ contains a pair $\langle a, b \rangle$ such that $b < a$. Thus, if congruences of the form $\mathrm{Cg}^{\mathbf{A}}(a, b)$, where $b \leq a$, are definable in V by a disjunction of principal congruence formulas Ψ, then we can write down a sentence Φ_1 such that

$$W \models \Phi_1$$

if and only if

$$\forall zu \; (u \leq z \to \text{“}\{\langle x, y \rangle : \Psi(x, y, z, u)\} \text{ is } \mathrm{Cg}^{\mathbf{A}}(z, u)\text{”})$$

holds in every $\mathbf{A} \in W$. Since the class of subdirect irreducibles of V is first-order definable, there is a sentence Θ such that, if $\mathbf{A} \in W$,

$$\mathbf{A} \models \Theta \quad \text{if and only if} \quad \mathbf{A} \in V_{SI}.$$

Let Φ_2 be a sentence asserting that the $\wedge$ reduct of an algebra is a meet–semilattice with 0 as the smallest element. Now, define Φ_3 to be the sentence

$$\Phi_1 \,\&\, \Phi_2 \,\&\, [\exists xy(x \neq y \,\&\, \forall zu\,(z < u \to \Psi(x, y, z, u))) \to \Theta].$$

It is easily seen that Φ_3 will be true in V.

Let Σ be an equational base of V. Since

$$\Sigma \models \Phi_3,$$

by the compactness theorem, there exists a finite subset Σ' of Σ such that

$$\Sigma' \models \Phi_3.$$

If $\mathbf{B}$ is any subdirectly irreducible algebra in the language of V, which satisfies Σ', it will also satisfy Φ_3, and, thus, it will be isomorphic to some subdirectly irreducible member of V. Hence, Σ' will be a finite equational basis for V. ∎

LEMMA 11. *Let V be a variety generated by a finite flat algebra and let $s(x, \bar{v})$, $t(x, \bar{w})$ be two terms in which x occurs explicitly. Then*

$$V \models \forall zu\bar{v}\bar{w}(u \leq z \to (\{s(z, \bar{v}), s(u, \bar{v})\} \cap \{t(z, \bar{w}), t(u, \bar{w})\} \neq \varnothing$$
$$\to s(u, \bar{v}) = t(u, \bar{w}))).$$

Proof. First, note that every subdirectly irreducible member of a variety generated by a finite flat algebra is itself a flat algebra (see, e.g., [11]). Also, if $p(x)$ is a unary polynomial of $\mathbf{B} \in V$, built from a term in which x occurs explicitly, then $p(0) = 0$, since 0 is an absorbing element for any algebra of V. Using these two facts it is straightforward to prove the validity of the sentence in every subdirectly irreducible algebra of V.

The condition

$$\{s(z, \bar{v}), s(u, \bar{v})\} \cap \{t(z, \bar{w}), t(u, \bar{w})\} \neq \varnothing$$

is equivalent to the formula

$$s(z, \bar{v}) = s(z, \bar{v}) \vee s(z, \bar{v}) = t(u, \bar{w}) \vee s(u, \bar{v}) = t(z, \bar{w}) \vee s(u, \bar{v}) = t(u, \bar{w}).$$

Hence, the original formula is equivalent to a conjunction of four universal Horn sentences and, since it is true in every member of V_{SI}, it will be true in every algebra of V. ∎

Fix a finite flat algebra $\mathbf{A}$ and consider the quantifier-free formulas of the forms

(0) $x = y$,

(1) $x = s(x, z) \,\&\, y = s(x, u)$,

(2) $x = s(y, u) \,\&\, y = s(y, z)$,

(3) $x = s(x, z) \,\&\, y = t(y, z) \,\&\, s(x, u) = t(y, u)$,

where s and t are terms which may contain additional parameters and in which z (resp. u) occurs explicitly. We also allow for the possibility that x (resp. y) does not occur in s (resp. t).

Let $\Phi(x, y, z, u) = \{\phi_i(x, y, z, u) : i < \omega\}$ be the collection of all principal congruence formulas whose quantifier-free parts are of the forms (1)–(3). Also, $\sigma_i(x, y, z, u, \bar{v})$ will denote the quantifier-free part of $\phi_i(x, y, z, u)$.

LEMMA 12. *Let $\mathbf{A}$ be a finite flat algebra, and, let $\sigma_i(x, y, z, u, \bar{v})$ and $\sigma_j(x, y, z, u, \bar{w})$ be such that*

$$\phi_i(x, y, z, u) = \exists \bar{v} \sigma_i(x, y, z, u, \bar{v}),$$
$$\phi_j(x, y, z, u) = \exists \bar{w} \sigma_j(x, y, z, u, \bar{w})$$

are both in $\Phi(x, y, z, u)$. Then there is $\sigma_k(x, y, z, u, \bar{v}, \bar{w})$ such that for every $\mathbf{B} \in V_{SI}(\mathbf{A})$ and all $a, b, c, d, e \in B$,

$$\mathbf{B} \models b \leq a \to \forall\, \bar{v}\bar{w}(\sigma_i(c, d, a, b, \bar{v}) \,\&\, \sigma_j(d, e, a, b, \bar{w})$$
$$\to \sigma_k(c, e, a, b, \bar{v}, \bar{w})).$$

Proof. The proof breaks into 16 cases, depending on the types of σ_i and σ_j. If the type of either formula is (0), the choice for σ_k is obvious.

Now, suppose that both σ_i and σ_j are of type (1); that is, we have

$$c = s(c, a, \bar{f}), \quad d = s(c, b, \bar{f}),$$
$$d = s'(d, a, \bar{f}'), \quad e = s'(d, b, \bar{f}'),$$

where $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b, c, d, e \in B$, $b \leq a$, and $\bar{f}$ and $\bar{f}'$ are tuples of elements of B, while s and s' are terms. Using Lemma 11, we get

$$e = s'(d, b, \bar{f}') = s(c, b, \bar{f}),$$

so σ_k can be chosen to be

$$x = s(x, z) \,\&\, y = s(x, u).$$

Next, suppose σ_i is of the type (1) while σ_j is of the type (2). Again, suppose $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b, c, d, e \in B$, $b \leq a$, and $\bar{f}$ and $\bar{f}'$ are tuples of elements of B, while s and s' are terms. Then

$$c = s(c, a, \bar{f}), \quad d = s(c, b, \bar{f}),$$
$$d = s'(e, b, \bar{f}'), \quad e = s'(e, a, \bar{f}').$$

Thus, we have σ_k of type (3):

$$x = s(x, z, \bar{v}) \,\&\, y = s'(y, z, \bar{w}) \,\&\, s(x, u, \bar{v}) = s'(y, u, \bar{w}).$$

In the next case, suppose σ_i is of type (1), and σ_j is of type (3). As before, we assume that $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b, c, d, e \in B$, $b \leq a$, and $\bar{f}$ and $\bar{f}'$ are tuples of elements of B, while s, s', and s'' are terms, so that

$$c = s(c, a, \bar{f}), \quad d = s(c, b, \bar{f}),$$
$$d = s'(d, a, \bar{f}'), \quad e = s''(e, a, \bar{f}'), \quad s'(d, b, \bar{f}') = s''(e, b, \bar{f}').$$

Using Lemma 11, we get

$$d = s(c, b, \bar{f}) = s'(d, b, \bar{f}') = s''(e, b, \bar{f}').$$

Hence, we have σ_k of type (3):

$$x = s(x, z, \bar{v}) \,\&\, y = s''(y, z, \bar{w}) \,\&\, s(x, u, \bar{v}) = s(y, u, \bar{w}).$$

The other six cases are handled in a similar fashion. The following table shows how σ_k depends on σ_i and σ_j:

	(1)	(2)	(3)
(1)	(1)	(3)	(3)
(2)	(0)	(2)	(2)
(3)	(1)	(3)	(3)

∎

We may now prove the following theorem.

THEOREM 13. *The collection $\Phi(x, y, z, u)$ of principal congruence formulas defines ordered principal congruences in $V(\mathbf{A})$ for every finite flat algebra $\mathbf{A}$.*

Proof. We first claim that $\Phi(x, y, z, u)$ defines ordered principal congruences in $V_{SI}(\mathbf{A})$. Let $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b \in B$, such that $b \leq a$, and let $\rho(a, b)$ be the binary relation on B defined by

$$\langle c, d\rangle \in \rho(a, b) \quad \text{if and only if} \quad \mathbf{B} \models \phi_i(c, d, a, b)$$

for some $\phi_i \in \Phi$.

Obviously, $\rho(a, b)$ is reflexive, symmetric, and compatible with all fundamental operations of **B**. By the preceding lemma, it is transitive. Clearly, $\mathrm{Cg}^{\mathbf{B}}(a, b)$ is contained in $\rho(a, b)$, since if $p(x)$ is a unary polynomial of **B**, $\langle p(a), p(b)\rangle \in \rho(a, b)$, and the latter equivalence relation is transitive. Conversely, $\rho(a, b) \subseteq \mathrm{Cg}^{\mathbf{B}}(a, b)$, for if $\mathbf{B} \models \phi_i(c, d, a, b)$, where $\phi_i \in \Phi$, then either there is $p(x) \in \mathrm{Pol}_1(\mathbf{B})$ such that

$$\{c, d\} = \{p(a), p(b)\}$$

or there are two unary polynomials p and q, and $e \in B$, so that

$$\{c, e\} = \{p(a), p(b)\} \quad \text{and} \quad \{e, d\} = \{q(a), q(b)\}.$$

Thus, Φ defines ordered principal congruences in V_{SI}.

To see that the same holds for the whole variety V, observe that in light of Lemma 12 it is possible to write down a collection of universal Horn formulas which express transitivity of the binary relation $\rho(a, b)$ defined as above, for every algebra $\mathbf{B} \in V$, and all $a, b \in V$ such that $b \leq a$. All other properties (reflexivity, symmetry, and compatibility) will lift automatically from the class of subdirect irreducibles to the whole of V. ∎

3. AXIOMATIZING THE CLASS OF SUBDIRECTLY IRREDUCIBLE ALGEBRAS IN $V(\mathbf{G}^{\wedge})$ WHEN G OMITS THE FOUR FORBIDDEN INDUCED SUBGRAPHS

From this point on, we are primarily interested in those finite graph flat algebras which omit every one of the four graphs $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$, and $\mathbf{P}_4$ as induced subgraphs. The main objective of this section will be to prove the following result:

THEOREM 14. *If* **A** *is a finite flat graph algebra which omits* $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$, *and* $\mathbf{P}_4$, $V_{SI}(\mathbf{A})$ *can be axiomatized by a single first-order sentence.*

DEFINITION 15. Let $\kappa, \lambda \geq 1$ be any cardinals. K_κ^0 will denote the complete looped graph on κ vertices, while $K_{\kappa,\lambda}$ denotes the complete bipartite graph with no loops whose blocks are of size κ and λ, respectively.

Let **A** be a finite flat graph algebra with the aforementioned property.

It was shown by Willard [11] that in $V(\mathbf{A})$, every subdirectly irreducible algebra is a simple flat graph algebra whose underlying graph is connected.

In what follows we will need to employ the notions of a subgraph and a direct product of graphs. Since there are several common such notions used in the literature, at this point we will specify those that we will work with.

By a subgraph we understand an induced subgraph in the usual graph-theoretic sense.

Given an indexed family of graphs $(\mathbf{G}_i\colon i \in I)$, where $\mathbf{G}_i = \langle V_i, E_i \rangle$, the vertex set of their direct product $\mathbf{G} = \prod_{i\in I} \mathbf{G}_i$ will be the set $V = \prod_{i \in I} V_i$ and, for $a, b \in V$, $\langle a, b\rangle \in E^{\mathbf{G}}$ if and only if $\langle a(i), b(i)\rangle \in E_i$ for every $i \in I$. (This is also referred to as the *categorical product* by graph theorists.)

PROPOSITION 16. *If* $\mathbf{A}$ *omits every one of* $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$, *and* $\mathbf{P}_4$ *as an induced subgraph, then the same is true of every* $\mathbf{B} \in V_{SI}(\mathbf{A})$.

Proof. We use the description of subdirect irreducibles in a variety generated by a finite flat algebra, as given in [11].

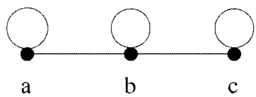
Assume $\mathbf{B} \in V_{SI}(\mathbf{A})$. Let A^+ and B^+ denote the underlying graphs of nonzero elements of A and B, respectively. Then, for some set I,

$$B^+ \leq (A^+)^I,$$

as graphs.

If $\mathbf{B}$ contains an induced subgraph isomorphic to $\mathbf{M}$, it is easily seen that the same is true of $\mathbf{A}$.

Now, if $\mathbf{B}$ contains an induced subgraph isomorphic to $\mathbf{L}_3$,

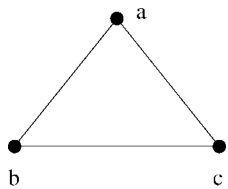


where $a, b, c \in (A^+)^I$, choose $i_0 \in I$ such that

$$a(i_0) \circ c(i_0) = 0.$$

Then $b(i_0) \notin \{a(i_0), c(i_0)\}$ and $a(i_0), b(i_0), c(i_0)$ induce a subgraph in A^+ isomorphic to $\mathbf{L}_3$.

Suppose B^+ contains a copy of $\mathbf{T}$,



where $a, b, c \in (A^+)^I$. Now choose $i_0 \in I$ so that

$$a(i_0) \circ a(i_0) = 0$$

in A^+. By analyzing different cases we arrive at the conclusion that either a copy of $\mathbf{T}$ or of $\mathbf{M}$ will be present in A^+.

Finally, if B^+ contains a copy of $\mathbf{P}_4$,

where $a, b, c, d \in (A^+)^I$, choose a coordinate $i_0 \in I$ for which

$$a(i_0) \circ d(i_0) = 0.$$

Then, the possibilities that $a(i_0) = c(i_0)$ or that $b(i_0) = d(i_0)$ can be ruled out immediately. If $a(i_0) = b(i_0)$ or $c(i_0) = d(i_0)$ or $b(i_0) = c(i_0)$, A^+ will contain $\mathbf{M}$ as an induced subgraph. In the case when all four of $a(i_0), b(i_0), c(i_0)$, and $d(i_0)$ are distinct, A^+ will contain $\mathbf{P}_4$ as an induced subgraph. ∎

Thus, if $\mathbf{B}$ is a subdirectly irreducible member of $V(\mathbf{A})$, $\mathbf{B}$ can belong only to one of the following classes of flat graph algebras:

1. $(K_\kappa^0)^\wedge$, where $\kappa \geq 1$.
2. $(K_{\kappa,\lambda})^\wedge$, where $\kappa, \lambda \geq 1$.
3. A flat graph algebra whose underlying graph consists of a single vertex without a loop.
4. A trivial flat graph algebra whose only element is 0.

However, the presence of $\mathbf{B}$ of type (1) or (2) in $V_{SI}(\mathbf{A})$, where $\kappa, \lambda \geq 2$, will be sufficient to describe "almost all" algebras in $V_{SI}(\mathbf{A})$. Namely, we have the following proposition.

PROPOSITION 17. (a) *If $\kappa \geq 2$, then $V((K_\kappa^0)^\wedge)$ contains every $(K_\mu^0)^\wedge$, where $\mu \geq 1$.*

(b) *If $\kappa \geq 2$, then $V((K_{\kappa,1})^\wedge)$ contains every $(K_{\mu,1})^\wedge$, where $\mu \geq 1$.*

(c) *If $\kappa, \lambda \geq 2$, then $V((K_{\kappa,\lambda})^\wedge)$ contains every $(K_{\mu,\nu})^\wedge$, where $\mu, \nu \geq 1$.*

Proof. (a) Let a, b be two distinct elements of $(K_\kappa^0)^\wedge$. Define $c^{(i)}$ $(0 \leq i < \mu)$ to be the μ sequence of elements from $(K_\kappa^0)^\wedge$, so that

$$c^{(i)}(j) = \begin{cases} a, & j \leq i, \\ b, & i < j. \end{cases}$$

Let $\mathbf{A}$ be the subalgebra of $((K_\kappa^0)^\wedge)^\mu$ generated by $c^{(i)}$ $(0 \leq i < \mu)$ and let θ be the smallest congruence of $\mathbf{A}$ which identifies all μ sequences in $\mathbf{A}$ which contain a 0 entry. The reader can check that $\mathbf{A}/\theta$ will be a flat graph algebra isomorphic to $(K_\mu^0)^\wedge$.

The proof of (b) is a slight modification of a more general proof for (c) and is therefore omitted.

(c) Let a, b be two distinct elements of one of the two bipartite blocks of $K_{\kappa,\lambda}$, and let c, d be two distinct elements of the other block. Let $\xi = \max(\mu, \nu)$. Define $e^{(i)}$ $(0 \leq i < \mu)$, a ξ sequence of elements from $K_{\kappa,\lambda}$, and $f^{(i)}$ $(0 \leq i < \nu)$, a ξ sequence of elements from $K_{\kappa,\lambda}$ as

$$e^{(i)}(j) = \begin{cases} a, & j \leq i, \\ b, & i < j; \end{cases}$$

$$f^{(i)}(j) = \begin{cases} c, & j \leq i, \\ d, & i < j. \end{cases}$$

Let $\mathbf{A}$ be the subalgebra of $((K_{\kappa,\lambda})^\wedge)^\xi$ generated by $e^{(i)}$ $(0 \leq i < \mu)$ and $f^{(i)}$ $(0 \leq i < \nu)$. Again, let θ be the congruence of $\mathbf{A}$ obtained by identifying all ξ sequences containing 0. Then $\mathbf{A}/\theta$ will be a graph flat algebra isomorphic to $(K_{\mu,\nu})^\wedge$. ∎

As an immediate consequence of the proposition, we obtain a proof of Theorem 14.

4. $V(\mathbf{G}^\wedge)$ HAS DEFINABLE ORDERED PRINCIPAL CONGRUENCES WHEN $\mathbf{G}$ OMITS THE FOUR FORBIDDEN INDUCED SUBGRAPHS

To show that every variety generated by a finite flat graph algebra which omits the four graphs must be finitely based, we show that the number of parameters required in the definition of quantifier-free formulas $\sigma_i(x, y, z, u, \bar{v})$ can be bounded. The number of such formulas will then be essentially finite, since the language in question is finite, and Φ can be reduced to a finite subset $\Phi' \subseteq \Phi$, which will define ordered principal congruences in V.

Let L' be the restriction of the language of flat graph algebras $L = \{\circ, \wedge, 0\}$ to the language $\{\circ, 0\}$. We show first that for every finite flat graph algebra $\mathbf{A}$ which omits $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, and $\mathbf{P}_4$, the unary L' polynomials depend on at most two parameters from A.

LEMMA 18. *Let $\mathbf{A}$ be a flat graph algebra, which omits $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, and $\mathbf{P}_4$, and let $V = V(\mathbf{A})$. Then, for every $(n+1)$-ary term $s(x, \bar{u})$ in L', where $n \geq 2$, there exists a ternary L' term $p_s(x, y, z)$ and two $(n+1)$-ary terms $t_1(x, \bar{u})$ and $t_2(x, \bar{u})$ such that*

$$V \models s(x, \bar{u}) = p_s(x, t_1(x, \bar{u}), t_2(x, \bar{u})).$$

Proof. We use the remarks made immediately after the proof of Proposition 16. The following identities are true in every subdirectly irreducible algebra in V (and, thus, in V):

$$
\begin{aligned}
(x \circ y) \circ z &= (x \circ z) \circ y, \\
(x \circ (y \circ z)) \circ u &= (x \circ u) \circ (y \circ z), \\
((x \circ y) \circ z) \circ u &= ((x \circ u) \circ y) \circ z, \\
((x \circ y) \circ z) \circ u &= (x \circ y) \circ (u \circ (x \circ z)).
\end{aligned}
$$

Using these identities, we prove, by induction on the number of occurrences of the operation symbol $\circ$ in a $(n+1)$-ary term $s(x, \bar{u})$, where $n \geq 3$, that the term $p_s(x, y, z)$ can be chosen to be one of

$$x, y, \quad x \circ y, \quad y \circ x, \quad z \circ (y \circ x), \quad z \circ (x \circ y), \quad (y \circ x) \circ z, \quad (x \circ y) \circ z,$$

while the two $(n+1)$-ary terms $t_1(x, \bar{u})$ and $t_2(x, \bar{u})$ will depend on $s(x, \bar{u})$.

The statement is clearly true if $s(x, \bar{u})$ contains no occurrences of $\circ$.

Suppose $s(x, \bar{u})$ is of the form

$$s(x, \bar{u}) = s'(x, \bar{u}) \circ s''(x, \bar{u}).$$

By induction hypothesis,

$$s'(x, \bar{u}) = p_{s'}(x, t_1'(x, \bar{u}), t_2'(x, \bar{u})).$$

for some $(n+1)-$ary terms $t_1'(x, \bar{u})$ and $t_2'(x, \bar{u})$ and a ternary term $p_{s'}(x, y, z)$ which has one of the eight forms displayed above.

The proof now breaks into eight cases depending on the particular form of $p_{s'}(x, y, z)$. We will provide the full argument for one of these cases and leave the rest as an exercise to the reader.

Suppose

$$p_{s'}(x, y, z) = (x \circ y) \circ z.$$

Then

$$
\begin{aligned}
s(x, \bar{u}) &= ((x \circ t_1'(x, \bar{u})) \circ t_2'(x, \bar{u})) \circ s''(x, \bar{u}) \\
&= (x \circ t_1'(x, \bar{u})) \circ (s''(x, \bar{u}) \circ (x \circ t_2'(x, \bar{u}))),
\end{aligned}
$$

where the second equality follows from the fourth equation displayed earlier in the proof.

Now, for $p_s(x, y, z)$ we can take the term $(x \circ y) \circ z$ and for $t_1(x, \bar{u})$, take the term $t_1'(x, \bar{u})$, while the term $t_2(x, \bar{u})$ can be taken to be $s''(x, \bar{u}) \circ (x \circ t_2'(x, \bar{u}))$. ∎

LEMMA 19. *Let $t(x, \bar{u})$ be an $(n+1)$-ary term in the language of flat graph algebras in which x occurs explicitly. Then there is an $(m+1)$-ary term $t'(x, \bar{v})$ in L' in which x occurs explicitly, where $m \leq n$ and $\bar{v}$ is a subtuple of $\bar{u}$, such that*

$$V \models t(x, \bar{u}) \leq t'(x, \bar{v}).$$

The proof of this lemma is by induction on the complexity of the term $t(x, \bar{u})$, and will be left to the reader.

Now that we have a bound on the number of parameters for L' polynomials, we can establish a similar bound for all L polynomials:

LEMMA 20. *Let $\mathbf{A}$ be a flat graph algebra, which omits $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, and $\mathbf{P}_4$, and let $V = V(\mathbf{A})$. Then, for every $(n+1)$-ary term $s(x, \bar{u})$ in L, where $n \geq 3$, there exists a 4-ary L term $q_s(x, y, z, w)$ and three $(n+1)$-ary terms $t_1(x, \bar{u})$, $t_2(x, \bar{u})$, and $t_3(x, \bar{u})$, such that*

$$V \models s(x, \bar{u}) = q_s(x, t_1(x, \bar{u}), t_2(x, \bar{u}), t_3(x, \bar{u})).$$

Proof. Let $s(x, \bar{u})$ be an $(n+1)$-ary term in L, where $n \geq 3$. By Lemma 20, there is an L' term s', which can be assumed to contain the same variables as s, such that

$$s(x, \bar{u}) = s(x, \bar{u}) \wedge s'(x, \bar{u}).$$

By Lemma 18, one can find a ternary term $p_{s'}(x, y, z)$ and two $(n+1)$-ary terms $t_1'(x, \bar{u})$ and $t_2'(x, \bar{u})$, so that

$$V \models s'(x, \bar{u}) = p_{s'}(x, t_1'(x, \bar{u}), t_2'(x, \bar{u})).$$

Then, for the 4-ary term $q_s(x, y, z, w)$ we can choose

$$w \wedge p_{s'}(x, y, z),$$

and for the terms $t_1(x, \bar{u})$, $t_2(x, \bar{u})$, and $t_3(x, \bar{u})$,

$$t_1(x, \bar{u}) = t_1'(x, \bar{u}),$$

$$t_2(x, \bar{u}) = t_2'(x, \bar{u}),$$

$$t_3(x, \bar{u}) = s(x, \bar{u}).$$

In that case,

$$V \models s(x, \bar{u}) = s(x, \bar{u}) \wedge s'(x, \bar{u}) = s(x, \bar{u}) \wedge p_{s'}(x, t_1'(x, \bar{u}), t_2'(x, \bar{u})).$$

∎

As an immediate corollary of Lemma 20, we deduce the following proposition.

PROPOSITION 21. *There is a finite subset $\Phi'(x, y, z, u) \subseteq \Phi(x, y, z, u)$, so that for every finite flat graph algebra $\mathbf{A}$, which omits $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, and $\mathbf{P}_4$,*

$$V(\mathbf{A}) \models \forall xyzu \left(\bigvee \Phi'(x, y, z, u) \longleftrightarrow \bigvee \Phi(x, y, z, u) \right).$$

In particular, $V(\mathbf{A})$ has definable ordered principal congruences.

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